

Singularly perturbed convection-diffusion boundary value problems with two small parameters using nonpolynomial spline technique

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Abstract In this paper, a new nonpolynomial cubic spline method is developed for solving two-parameter singularly perturbed boundary value problems. Convergence analysis is briefly discussed. Numerical examples and computational results illustrate and guarantee a higher accuracy by this technique. Comparisons are made to confirm the reliability and accuracy of the proposed technique.

Keywords Singular perturbation · Nonpolynomial cubic spline · Convergence analysis · Boundary value problem · Convection-diffusion

Introduction

We consider the two-parameter singularly perturbed convection-diffusion boundary value problems of the form:

$$Ly(x) \equiv -\epsilon y''(x) + \mu p(x)y'(x) + f(x)y(x) = g(x), \quad x \in (a, b) \quad (1.1)$$

subjected to the boundary conditions:

$$y(0) = \alpha_0, \quad y(1) = \alpha_1.$$

with two small positive parameters $0 < \epsilon \ll 1$, $0 < \mu \ll 1$, where $p(x)$, $f(x)$, and $g(x)$ are sufficiently smooth real

valued functions with $p(x) \geq p^* > 0$, $f(x) \geq f^* > 0$, and $g(x) \geq g^* > 0$ for $x \in (a, b)$. Under these assumptions, problem (1.1) is characterized into two cases:

1. For $\mu = 0$, problem (1.1) becomes reaction-diffusion problem.
2. For $\mu = 1$, problem (1.1) becomes convection-diffusion problem.

This type of problem arises in the fields like engineering, mathematical physics, and in many areas of applied mathematics. We often come across boundary value problems in which one or small positive parameter multiplies with the derivatives. A large number of research papers have been found in the literature for single parameter convection-diffusion and reaction-diffusion problems [2, 8, 9, 12, 16]. However, only a very few authors have discussed two-parameter singularly perturbed boundary value problems [4, 6, 7, 10, 11, 14, 16, 18–20]. The nature of two parameters is asymptotically examined by O' Malley [14]. Different numerical methods have been proposed by various authors for two-parameter singularly perturbed problems such as exponentially fitted cubic spline method [7], finite difference, finite element, and B-spline collocation method [6, 11], Haar wavelet method [16], and exponential spline technique [18]. For more information about SPPs, readers are referred to books [13, 15] and references therein.

In this paper, we introduce a new nonpolynomial cubic spline method as an alternative to existing methods. The paper is organised into five sections. In Sect. 2, we give a brief derivation of nonpolynomial parameters cubic spline. In Sect. 3, we presented the formulation of the method. Convergence analysis is briefly discussed in Sect. 4. Finally, in Sect. 5, numerical examples and comparison

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with the existing methods are given that demonstrate the practical applicability and superiority of the proposed method.

Nonpolynomial spline function

We consider a uniform mesh Δ with nodal points x_i on $[a, b]$, such that $\Delta : a = x_0 < x_1 < x_2 < \dots < x_n - 1 < x_n = b$, where $x_i = a + ih, i = 0, 1, \dots, n$, and $h = \frac{(b-a)}{n}$. A nonpolynomial spline function $S_\Delta(x)$ of class $C^2[a, b]$ which interpolates $y(x)$ at mesh points $x_i, i = 0(1)n$ depends on a parameter k , if we take $k \rightarrow 0$, then it reduces to ordinary cubic spline in $[a, b]$.

For each segment $[x_i, x_{i+1}], i = 0, 1, 2, \dots, n - 1$, we consider the nonpolynomial cubic spline $S_\Delta(x)$ of the form:

$$S_\Delta(x) = a_i \sin k(x - x_i) + b_i \cos k(x - x_i) + c_i e^{k(x-x_i)} + d_i e^{-k(x-x_i)}, \quad i = 0, 1, \dots, n, \tag{2.1}$$

where a_i, b_i, c_i , and d_i are unknown coefficients and k is a free parameter which will be used to raise the accuracy of the method.

Let $y(x)$ be the exact solution and y_i be an approximation to $y(x_i)$, obtained by the segment $S_i(x)$ of the mixed splines function passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}) . To determine the coefficients of Eq. (2.1) in terms of $y_i, y_{i+1}, M_i, M_{i+1}$, we first define:

$$\left. \begin{aligned} S_\Delta(x_i) &= y_i, & S_\Delta(x_{i+1}) &= y_{i+1}, \\ S''_\Delta(x_i) &= M_i, & S''_\Delta(x_{i+1}) &= M_{i+1}. \end{aligned} \right\} \tag{2.2}$$

We obtain via a long but straightforward calculation

$$\begin{aligned} a_i &= \frac{(k^2 y_{i+1} - M_{i+1}) - \cos \theta (k^2 y_i - M_i)}{2k^2 \sin \theta}, \\ b_i &= \frac{(k^2 y_i - M_i)}{2k^2}, \\ c_i &= \frac{e^\theta (k^2 y_{i+1} + M_{i+1}) - (k^2 y_i + M_i)}{2k^2 (e^{2\theta} - 1)}, \\ d_i &= \frac{e^{2\theta} (k^2 y_i + M_i) - e^\theta (k^2 y_{i+1} + M_{i+1})}{2k^2 (e^{2\theta} - 1)}, \\ \theta &= kh \quad \text{and} \quad i = 0(1)n - 1. \end{aligned}$$

Using the continuity of the first derivative at the point $x = x_i$, we obtain the following tridiagonal system for $i = 1, 2, \dots, n - 1$:

$$y_{i-1} + \gamma y_i + y_{i+1} = h^2 (\alpha M_{i-1} + \beta M_i + \alpha M_{i+1}), \tag{2.3}$$

where

$$\begin{aligned} \alpha &= \frac{(e^{2\theta} - 2e^\theta \sin \theta - 1)}{\theta^2 (e^{2\theta} + 2e^\theta \sin \theta - 1)}, \\ \beta &= 2 \frac{[e^{2\theta} (\sin \theta - \cos \theta) - (\sin \theta + \cos \theta)]}{\theta^2 (e^{2\theta} + 2e^\theta \sin \theta - 1)}, \\ \gamma &= -2 \frac{[e^{2\theta} (\sin \theta + \cos \theta) + (\sin \theta - \cos \theta)]}{\theta^2 (e^{2\theta} + 2e^\theta \sin \theta - 1)}. \end{aligned}$$

If $\theta \rightarrow 0$, then $(\alpha, \beta, \gamma) \rightarrow (\frac{1}{6}, \frac{4}{6}, -2)$, and then spline defined by (2.3) reduces to an ordinary cubic spline relation [13]:

$$(y_{i-1} - 2y_i + y_{i+1}) = \frac{h^2}{6} (M_{i-1} + 4M_i + M_{i+1}). \tag{2.4}$$

The relation (2.3) gives $(n - 1)$ linear algebraic equations in $(n - 1)$ unknowns $y_i, i = 1, 2, \dots, n - 1$.

The method

At the grid point x_i , the proposed two-parameter singularly perturbed boundary value problem (1.1) can be discretized as follows:

$$-\epsilon y''(x_i) + \mu p(x_i) y'(x_i) + f(x_i) y(x_i) = g(x_i). \tag{3.1}$$

Using spline's second derivative, we have

$$\begin{aligned} M_i &= \frac{\mu p_i y'_i + f_i y_i - g_i}{\epsilon}, \\ M_{i-1} &= \frac{\mu p_{i-1} y'_{i-1} + f_{i-1} y_{i-1} - g_{i-1}}{\epsilon}, \\ M_{i+1} &= \frac{\mu p_{i+1} y'_{i+1} + f_{i+1} y_{i+1} - g_{i+1}}{\epsilon}, \end{aligned}$$

where

$$\begin{aligned} y'_i &= \frac{y_{i+1} - y_{i-1}}{2h}, \\ y'_{i-1} &= \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}, \quad y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}, \end{aligned}$$

$p_i = p(x_i), f_i = f(x_i)$ and $g_i = g(x_i)$.

Substituting the values of $M_j (j = i, i \pm 1)$ in Eq. (2.3), we have

$$\begin{aligned} &\left[-\epsilon + \frac{\mu h}{2} (-3\alpha p_{i-1} - \beta p_i + \alpha p_{i+1}) + h^2 \alpha f_{i-1} \right] y_{i-1} \\ &+ \left[-\gamma \epsilon + \frac{\mu h}{2} (4\alpha p_{i-1} - 4\alpha p_{i+1}) + h^2 \beta f_i \right] y_i \\ &+ \left[-\epsilon + \frac{\mu h}{2} (-\alpha p_{i-1} + \beta p_i + 3\alpha p_{i+1}) + h^2 \alpha f_{i+1} \right] y_{i+1} \\ &= h^2 (\alpha g_{i-1} + \beta g_i + \alpha g_{i+1}), \quad i = 1(1)n - 1. \end{aligned} \tag{3.2}$$

Finally, we arrive at the following system:

$$\left\{ \begin{array}{l}
 \left[-\gamma\epsilon + \frac{\mu h}{2} V_1 + h^2 \beta f_1 \right] y_1 + \left[-\epsilon + \frac{\mu h}{2} W_1 + h^2 \alpha f_2 \right] y_2 \\
 = h^2 [\alpha(g_0 - f_0 \alpha_0) + \beta g_1 + \alpha g_2] + \epsilon \alpha_0 - \frac{\mu h}{2} \alpha_0 U_1, \quad i = 1, \\
 \left[-\epsilon + \frac{\mu h}{2} U_i + h^2 \alpha f_{i-1} \right] y_{i-1} + \left[-\gamma\epsilon + \frac{\mu h}{2} V_i + h^2 \beta f_i \right] y_i + \left[-\epsilon + \frac{\mu h}{2} W_i + h^2 \alpha f_{i+1} \right] y_{i+1} \\
 = h^2 (\alpha g_{i-1} + \beta g_i + \alpha g_{i+1}), \quad 2 \leq i \leq n - 2, \\
 \left[-\epsilon + \frac{\mu h}{2} U_{n-1} + h^2 \alpha f_{n-2} \right] y_{n-2} + \left[-\gamma\epsilon + \frac{\mu h}{2} V_{n-1} + h^2 \beta f_{n-1} \right] y_{n-1} \\
 = h^2 [\alpha(g_n - f_n \alpha_n) + \alpha g_{n-2} + \beta g_{n-1}] + \epsilon \alpha_1 - \frac{\mu h}{2} \alpha_n W_{n-1}, \quad i = n - 1,
 \end{array} \right. \tag{3.3}$$

where

$$U_i = (-3\alpha p_{i-1} - \beta p_i + \alpha p_{i+1}), \quad V_i = (4\alpha p_{i-1} - 4\alpha p_{i+1}), \\
 W_i = (-\alpha p_{i-1} + \beta p_i + 3\alpha p_{i+1}), \quad i = 1(1)n - 1.$$

Convergence analysis

In this section, we investigate the convergence analysis of the proposed method. For this, let $Y = y(x_i), \bar{Y} = (y_i), C = (c_i), T = (t_i), E = (e_i) = Y - \bar{Y}, i = 1, 2, \dots, n - 1$ be an exact column vectors, where $Y, \bar{Y}, T,$ and E are exact, approximate, local truncation error, and discretization error, respectively.

We can write the standard matrix equation for the method developed in the following form:

$$M\bar{Y} = C, \tag{4.1}$$

where M is a matrix of order $(n - 1)$ with

$$M = (A_0 + A_1 + h^2 A_2 F). \tag{4.2}$$

The tridiagonal matrices $A_0, A_1,$ and A_2 have the form:

$$A_0 = \begin{bmatrix} -\gamma\epsilon & -\epsilon & & & & \\ -\epsilon & -\gamma\epsilon & -\epsilon & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & -\epsilon & -\gamma\epsilon & -\epsilon \\ & & & & & -\epsilon & -\gamma\epsilon & \end{bmatrix}, \tag{4.3}$$

$$A_1 = \begin{bmatrix} \frac{\mu h}{2} V_1 & \frac{\mu h}{2} W_1 & & & & \\ \frac{\mu h}{2} U_2 & \frac{\mu h}{2} V_2 & \frac{\mu h}{2} W_2 & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \frac{\mu h}{2} U_{n-2} & \frac{\mu h}{2} V_{n-2} & \frac{\mu h}{2} W_{n-2} \\ & & & & \frac{\mu h}{2} U_{n-1} & \frac{\mu h}{2} V_{n-1} & \end{bmatrix}, \tag{4.4}$$

From Eqs. (4.9), (4.13), and (4.14), we have

$$e_i = \sum_{k=1}^{n-1} m_{k,i}^{-1} T_i(h), \quad k = 1, 2, \dots, n-1 \tag{4.18}$$

and therefore

$$|e_i| \leq \frac{Kh^2}{|Q_{i_0}|}, \quad i = 1, 2, \dots, n-1, \tag{4.19}$$

where K is a constant independent of h . It follows that $\|E\| = O(h^2)$.

However, for the choice of parameters, $\alpha = 1/12, \beta = 10/12$, and $\gamma = -2$,

$$|e_i| \leq \frac{Kh^4}{|Q_{i_0}|}, \quad i = 1, 2, \dots, n-1, \tag{4.20}$$

where K is a constant independent of h . It follows that $\|E\| = O(h^4)$.

We summarize the above result in the following theorem:

Theorem 4.1 *Let $y(x)$ be the exact solution of two-parameter singularly perturbed boundary value problem (1.1) and let y_i be the numerical solution obtained from the difference scheme (4.1). Then, for sufficiently small h , scheme gives a second-order convergent solution for any arbitrary choice of α and β with $\gamma = -2$ and a fourth-order convergent solution for $\alpha = 1/12, \beta = 10/12$, and $\gamma = -2$.*

Numerical examples

To test the viability of the proposed method based on nonpolynomial cubic spline, two numerical examples are considered. All the computations were performed using MATLAB. We also compare our method with the existing methods which shown improvement.

Example 1 Consider the following two-parameter singularly perturbed boundary value problem, which is discussed in [12, 19]:

$$-\epsilon y'' + \mu y' + y = 1, \quad x \in (0, 1), \tag{5.1}$$

subjected to the boundary conditions:

$$y(0) = 0, \quad y(1) = 0. \tag{5.2}$$

The exact solution of the above problem is

$$y(x) = \frac{(e^{\lambda_2} - 1)e^{\lambda_1 x}}{e^{\lambda_1} - e^{\lambda_2}} + \frac{(1 - e^{\lambda_1})e^{\lambda_2 x}}{e^{\lambda_1} - e^{\lambda_2}} + 1, \tag{5.3}$$

where

$$\lambda_1 = \frac{1 + \sqrt{1 + 4\epsilon}}{2\epsilon}, \quad \lambda_2 = \frac{1 - \sqrt{1 + 4\epsilon}}{2\epsilon}.$$

Example 2 Consider the following two-parameter singularly perturbed boundary value problem, which is discussed in [6, 16, 19]:

$$-\epsilon y'' + \mu y' + y = \cos(\pi x), \quad x \in (0, 1), \tag{5.4}$$

subjected to the boundary conditions:

$$y(0) = 0, \quad y(1) = 0. \tag{5.5}$$

The exact solution of the above problem is

$$u(x) = \rho_1 \cos(\pi x) + \rho_2 \sin(\pi x) + \psi_1 e^{\lambda_1 x} + \psi_2 e^{-\lambda_2(1-x)}, \tag{5.6}$$

where

$$\rho_1 = \frac{\epsilon \pi^2 + 1}{\epsilon^2 \pi^2 + (\epsilon \pi^2 + 1)^2}, \quad \rho_2 = \frac{\epsilon \pi}{\epsilon^2 \pi^2 + (\epsilon \pi^2 + 1)^2},$$

$$\psi_1 = -\rho_1 \frac{1 + e^{-\lambda_2}}{1 - e^{\lambda_1 - \lambda_2}}, \quad \psi_2 = \rho_1 \frac{1 + e^{\lambda_1}}{1 - e^{\lambda_1 - \lambda_2}},$$

$$\lambda_1 = \frac{\mu - \sqrt{\mu^2 + 4\epsilon}}{2\epsilon}, \quad \lambda_2 = \frac{\mu + \sqrt{\mu^2 + 4\epsilon}}{2\epsilon}.$$

The numerical results corresponding to the Examples 1 and 2 are briefly summarized in Tables 1, 2, 3, and 4, and Figs. 1, 2, 3, and 4. Comparison with other existing methods are also listed in Tables 1, 2, 3 and 4. These tables show that method is more accurate than the existing methods.

Table 1 Comparison of pointwise errors, Example 1

x ↓	ε=0.1, μ=1, n=32		ε=0.1, μ=1, n=128	
	Lin et al. [12]	Our method	Lin et al. [12]	Our method
1/16	2.74(-2)	3.27(-6)	6.8(-3)	1.15(-6)
2/16	2.59(-2)	5.44(-6)	6.4(-3)	2.18(-6)
4/16	2.30(-2)	3.98(-6)	5.7(-3)	3.72(-6)
6/16	2.04(-2)	1.61(-5)	5.0(-3)	3.92(-6)
12/16	2.50(-2)	8.52(-4)	4.0(-4)	4.45(-5)
14/16	3.30(-2)	1.70(-3)	9.4(-3)	9.61(-5)

Table 2 Comparison of pointwise errors, Example 1

$x \downarrow$	$\epsilon=0.01, \mu=1, n=32$		$\epsilon=0.01, \mu=1, n=128$	
	Lin et al. [12]	Our method	Lin et al. [12]	Our method
1/16	2.95(-2)	4.55(-6)	7.3(-3)	2.84(-7)
2/16	2.78(-2)	8.55(-6)	6.9(-3)	5.35(-7)
4/16	2.45(-2)	1.51(-5)	6.1(-3)	9.45(-7)
6/16	2.17(-2)	2.00(-5)	5.4(-3)	1.25(-6)
12/16	1.50(-2)	3.07(-5)	3.7(-3)	1.73(-6)
14/16	1.29(-2)	1.41(-3)	3.3(-3)	7.31(-7)

Table 3 Comparison of maximum absolute errors, Example 2

$\mu \downarrow$	$\epsilon=10^{-2}, n=128$			
	Kadalbajoo et al. [6]	Zahra et al. [19]	Pandit et al. [16]	Our method
10^{-3}	8.3832(-5)	4.1924(-5)	4.2303(-5)	6.0243(-6)
10^{-4}	8.2686(-5)	4.1296(-5)	4.1318(-5)	6.1827(-7)
10^{-5}	8.2572(-5)	4.1232(-5)	4.1220(-5)	1.1455(-7)
10^{-6}	8.2561(-5)	4.1226(-5)	4.1210(-5)	7.2269(-8)
10^{-7}	8.25596(-5)	4.1225(-5)	4.1209(-5)	6.8266(-8)

Table 4 Comparison of maximum absolute errors, Example 2

$\mu \downarrow$	$\epsilon=10^{-4}, n=128$			
	Kadalbajoo et al. [6]	Zahra et al. [19]	Pandit et al. [16]	Our method
10^{-3}	9.4446(-3)	4.7598(-3)	5.1964(-3)	6.2154(-3)
10^{-4}	9.0436(-3)	4.2856(-3)	4.1710(-3)	1.8330(-3)
10^{-5}	9.0036(-3)	4.2295(-3)	4.0754(-3)	1.1412(-3)
10^{-6}	8.9996(-3)	4.2238(-3)	4.0659(-3)	1.3699(-3)
10^{-7}	8.9992(-3)	4.2232(-3)	4.0650(-3)	1.3656(-3)

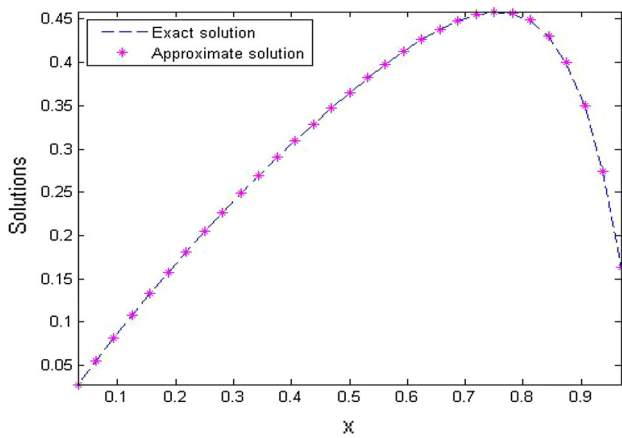


Fig. 1 Physical behaviour of numerical solution of Example 1 for $\epsilon = 0.1, \mu = 1, \text{ and } n = 32$

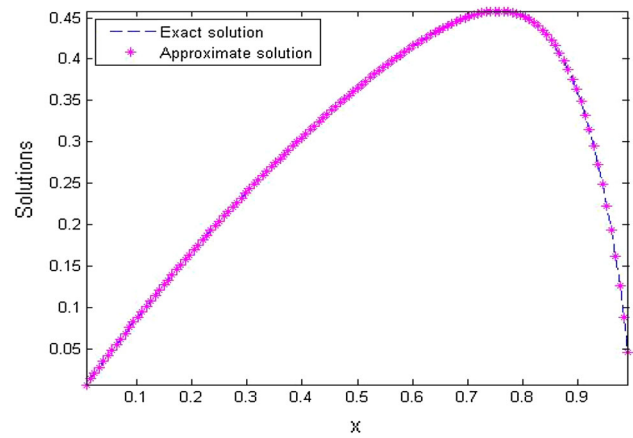


Fig. 2 Physical behaviour of numerical solution of Example 1 for $\epsilon = 0.1, \mu = 1, \text{ and } n = 128$

Tables 1, 2 show the pointwise errors at different values of n and for small values of ϵ . Tables 3, 4 show the maximum absolute errors of the Example 2 for different values of ϵ and μ . Figures 1, 2 compare the exact and approximate

solutions of Example 1 for $\epsilon = 0.1$ and $\mu = 1$, while Figs. 3 and 4 report the exact and approximate solutions of Example 2 for different values of ϵ and μ .

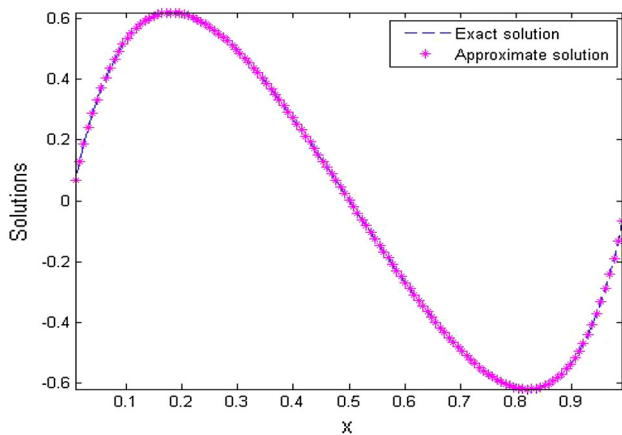


Fig. 3 Physical behaviour of numerical solution of Example 2 for $\epsilon = 10^{-2}$, $\mu = 10^{-3}$, and $n = 128$

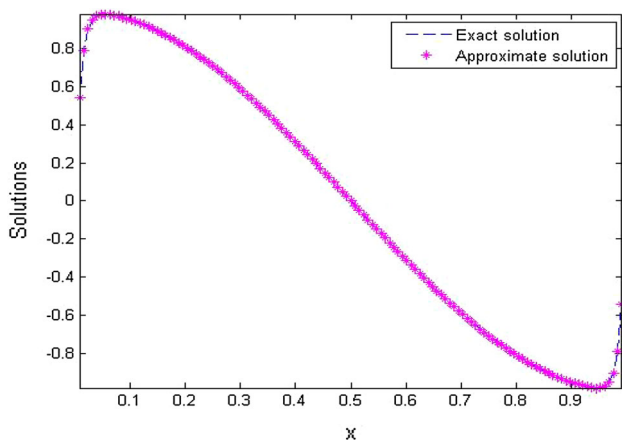


Fig. 4 Physical behaviour of numerical solution of Example 2 for $\epsilon = 10^{-4}$, $\mu = 10^{-5}$, and $n = 128$

Concluding remarks

In this paper, nonpolynomial cubic spline function is used for finding the numerical solution of two-parameter convection-diffusion singularly perturbed boundary value problems. The computations associated with the examples discussed above were performed using MATLAB. The proposed method is computationally efficient and the algorithm can be easily implemented on a computer. Comparison of the method is also depicted through Tables 1, 2, 3, and 4 which shown that our methods perform better in the sense of accuracy and applicability. The solution profiles for the considered examples for different values of ϵ and μ are given in Figs. 1, 2, 3, and 4.

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